Math 2040C Week 5

Null space and Ranges

Defn 3.12, 3.17 Let $T \in \mathcal{L}(V,W)$. Define <u>null space of T</u> and <u>range of T</u> to be

$$\text{null } T = \{v \in V : T(v) = \vec{0}_W\}$$

$$\text{range } T = \{T(v) \in W : v \in V\}$$

<u>Other notations</u>:

null T = ker T = Kernel of T

<u>eg</u> $I : V \to V$ has null $I = \{\vec{0}\}$, range $I = V$

$T_0 : V \to V$ has null $T_0 = V$, range $T_0 = \{\vec{0}\}$

<u>eg</u> $D : \mathcal{L}(P_n(\mathbb{R}), P_n(\mathbb{R}))$, $Df = f'$

null $T = \{c : c \in \mathbb{R}\} = \text{span}\{1\}$

range $T = P_{n-1}(\mathbb{R}) = \text{span}\{1, x, \cdots, x^{n-1}\}$

Prop 3.14, 3.19

Suppose $T \in \mathcal{L}(V,W)$. Then

null $T \subseteq V$ and range $T \subseteq W$ are subspace

We first prove for null T

① $T(\vec{0}_V) = \vec{0}_W \Rightarrow \vec{0}_V \in \text{null } T$

② Suppose $u, v \in$ null T. Then

$T(u+v) = T(u) + T(v) = \vec{0}_W + \vec{0}_W = \vec{0}_W$

$\therefore u+v \in$ null T

③ Suppose $v \in$ null T, $\lambda \in \mathbb{F}$, then

$T(\lambda v) = \lambda T(v) = \lambda \vec{0}_W = \vec{0}_W$

$\therefore \lambda u \in$ null T

$\therefore$ null T is a subspace of V

For range T :

① $\vec{0}_V \in V \Rightarrow \vec{0}_W = T(\vec{0}_V) \in \text{range } T$

② Suppose $w_1, w_2 \in \text{range } T$. Then $\exists\, v_1, v_2 \in V$

such that $w_1 = T(v_1)$, $w_2 = T(v_2)$.

$$\therefore\ w_1 + w_2 = T(v_1) + T(v_2)$$
$$= T(v_1 + v_2) \in \text{range } T$$

③ Suppose $w \in \text{range } T$, $\lambda \in \mathbb{F}$. Then

$\exists\, v \in V$ such that $w = T(v)$.

$$\therefore\ \lambda w = \lambda T(v) = T(\lambda v) \in \text{range } T$$

Hence, range T is a subspace of W ▨

<u>eg</u> (Review) Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix}\right) = A \begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & 3 \\ 2 & 4 & 3 & 4 \end{bmatrix}$$

We want to find basis of null T and range T

Note $\begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} \in \text{null } T \Leftrightarrow A \begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 0 \\ 1 & 2 & 2 & 3 & 0 \\ 2 & 4 & 3 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} \boxed{1} & 2 & 0 & -1 & 0 \\ 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \ (*)$$

pivot free

Let $x_2 = s$, $x_4 = t$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s+t \\ s \\ -2t \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore$ Basis for null T : $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

$(*)$ has pivot on 1st and 3rd column.

Corresponding columns of A form a basis of range T

$\therefore$ Basis of range T : $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$

Prop Suppose $T \in \mathcal{L}(V,W)$. Then

① T is injective $\Leftrightarrow$ $\text{null}\, T = \{0_V\}$

② T is surjective $\Leftrightarrow$ $\text{range}\, T = W$

Pf ② follows from definition

For pf of ①,

($\Rightarrow$) Suppose T is injective

$T(\vec{0_V}) = \vec{0_W} \Rightarrow \{\vec{0_V}\} \subseteq \text{null}\, T$

If $v \in \text{null}\, T$, then $T(v) = \vec{0_W} = T(\vec{0_V})$

T is injective $\Rightarrow$ $v = \vec{0_V}$

$\Rightarrow \text{null}\, T \subseteq \{\vec{0_V}\}$

$\therefore \text{null}\, T = \{\vec{0_V}\}$

($\Leftarrow$) Suppose $\text{null}\, T = \{0_V\}$

If $u, v \in V$ and $T(u) = T(v)$, then

$T(u-v) = T(u) - T(v) = \vec{0_W}$

$\Rightarrow u - v \in \text{null}\, T = \{0_V\}$

$\Rightarrow u - v = \vec{0_V} \Rightarrow u = v$

$\therefore T$ is injective

Prop Let $T \in \mathcal{L}(V,W)$. Then

① If $V = \text{span}\{v_1, \cdots, v_n\}$, then

$\text{range}\, T = \text{span}\{T(v_1), \cdots, T(v_n)\}$

② If $v_1, \cdots, v_n \in V$ are lin. dept,

Then $T(v_1), \cdots, T(v_n) \in W$ are lin. dept.

③ If $v_1, \cdots, v_n \in V$ are lin. indept, T is injective,

Then $T(v_1), \cdots, T(v_n) \in W$ are lin indept.

Pf Exercise

Thm 3.22 (Fundamental Theorem of Linear Maps)

Suppose $T \in \mathcal{L}(V,W)$, $\dim V < \infty$

Then $\dim \operatorname{range} T < \infty$ and

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

Rmk No assumption on $\dim W < \infty$

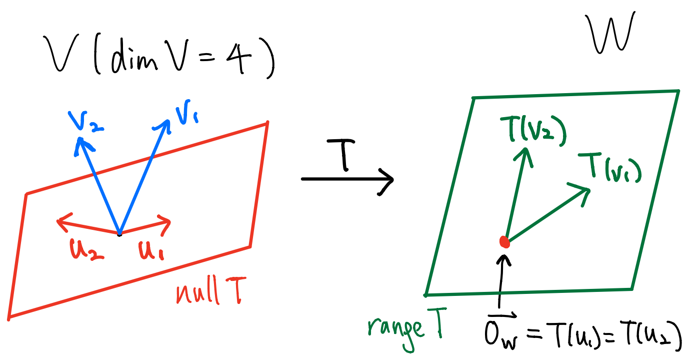
Pf Let $S' = \{u_1, \cdots, u_m\}$ be a basis of $\operatorname{null} T$

Extend it to a basis

$$S = \{u_1, \cdots, u_m, v_1, \cdots, v_n\} \text{ of } V$$

We want to show

$$S'' = \{T(v_1), \cdots, T(v_n)\} \text{ is a basis of } \operatorname{range} T$$



We first show that $\operatorname{span} S'' = \operatorname{range} T$

Clearly, $T(v_i) \in \operatorname{range} T \quad \forall i = 1, 2, \cdots, n$

$\Rightarrow \operatorname{span} S'' \subseteq \operatorname{range} T$

To show $\operatorname{range} T \subseteq \operatorname{span} S''$, let $w \in \operatorname{range} T$.

Then $w = T(v)$ for some $v \in V$

S is a basis of $V \Rightarrow \exists\, a_1, \cdots, a_m, b_1, \cdots, b_n \in \mathbb{F}$

s.t. $v = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n$

$$= \sum a_i u_i + \sum b_j v_j$$

$$\begin{aligned}
\therefore\ w = T(v) &= T\left(\sum a_i u_i + \sum b_j v_j\right) \\
&= \sum a_i T(u_i) + \sum b_j T v_j) \\
&= \sum a_i \vec{0}_W + \sum b_j T(v_j) \\
&= \sum b_j T(v_j) \in \operatorname{span} S''
\end{aligned}$$

$\Rightarrow \operatorname{range} T \subseteq \operatorname{span} S''$

$\therefore \operatorname{span} S'' = \operatorname{range} T$

Next, we show S'' is lin. indept.

Suppose $c_1T(v_1)+\cdots+c_nT(v_n)=\vec{0}_W$

Then $T(c_1v_1+\cdots+c_nv_n)=\vec{0}_W$

$\Rightarrow c_1v_1+\cdots+c_nv_n \in \text{null } T$

S' is a basis of null T

$\Rightarrow \exists\, d_1,\cdots,d_m \in \mathbb{F}$ such that

$$c_1v_1+\cdots+c_nv_n = d_1u_1+\cdots+d_mu_m$$

S is lin. indept $\Rightarrow$ all $c_j, d_i = 0$

$\Rightarrow S''$ is lin. indept

$\Rightarrow S''$ is a basis of range T

$$\begin{aligned}\therefore \dim V &= |S| = m+n \\ &= |S'|+|S''| \\ &= \dim \text{null}(T) + \dim \text{range } T\end{aligned}$$

Prop 3.23, 3.24

Let V, W are finite dim. and $T \in \mathcal{L}(V,W)$

1. If $\dim V > \dim W$, then T is not injective
2. If $\dim V < \dim W$, then T is not surjective

Rmk Injectivity / Surjectivity of linear maps are used to compare sizes of vector spaces.

The proposition means that dimension measures the size of vector spaces:

Higher dimensional vector spaces are bigger

Compare:

Fact There is a bijection between $\mathbb{R}^2$ and $\mathbb{R}$

$\therefore$ $\mathbb{R}^2$ and $\mathbb{R}$ have same "size" as SETS

Pf The pf makes use of the formula

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

① If $\dim V > \dim W$, then

$$\begin{aligned}\dim \operatorname{null} T &= \dim V - \dim \operatorname{range} T \\ &\geq \dim V - \dim W > 0\end{aligned}$$

$\because$ range T is a subspace of W

$\Rightarrow \operatorname{null} T \neq \{0\}$

$\Rightarrow T$ is not injective

② If $\dim V < \dim W$, then

$$\begin{aligned}\dim \operatorname{range} T &= \dim V - \dim \operatorname{null} T \\ &< \dim W - \dim \operatorname{null} T \\ &< \dim W\end{aligned}$$

$\Rightarrow \operatorname{range} T \neq W$

$\Rightarrow T$ is not surjective

eg Let $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be linear, defined by

$$T(f(x)) = 2f'(x) + \int_0^x 3f(t)\,dt$$

Is T injective? surjective?

Sol $\dim P_2(\mathbb{R}) = 3 < 4 = \dim P_3(\mathbb{R})$

$\therefore T$ is not surjective.

$P_2(\mathbb{R}) = \operatorname{span}\{1, x, x^2\}$

$$\begin{aligned}\Rightarrow \operatorname{range} T &= \operatorname{span}\{T(1), T(x), T(x^2)\} \\ &= \operatorname{span}\{\underbrace{3x,\ 2 + \tfrac{3}{2}x^2,\ 4x + x^3}_{\text{different degree} \Rightarrow \text{lin indept} \Rightarrow \text{basis}}\}\end{aligned}$$

$\therefore \dim \operatorname{null} T = \dim P_2(\mathbb{R}) - \dim \operatorname{range} T = 3 - 3 = 0$

$\therefore \operatorname{null} T = \{0_V\}$ and T is injective.

Prop Suppose $\dim V = \dim W$ is finite, $T \in \mathcal{L}(V, W)$

Then T is injective $\Longleftrightarrow$ T is surjective $\Longleftrightarrow$ T is bijective

Ex Prove

Rmk If $V = W$ and $\dim V = \infty$, then

T is injective or surjective $\not\Rightarrow$ T is bijective

Relation to systems of linear equations

A system of m lin. eqn. of n variables

$$(*) \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \quad \ddots \quad \vdots \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$\Leftrightarrow$ $A\vec{x} = \vec{b}$, where $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \in M_{m\times n}(\mathbb{F})$$

<u>Ex</u> Verify $T: \mathbb{F}^n \to \mathbb{F}^m$, $T(\vec{x}) = A\vec{x}$, is linear

Solve $(*)$ means finding $\vec{x} \in \mathbb{F}^n$ s.t. $A\vec{x} = \vec{b}$

$(*)$ has solution $\Leftrightarrow$ $\vec{b} \in \operatorname{range} T$

$$= \operatorname{span}\{T(e_1), \cdots, T(e_n)\}$$

$$= \operatorname{span}\left\{\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \cdots \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}\right\}$$

$$= \text{column space of } A$$

If solution exists,

Solution is unique $\Leftrightarrow$ T is injective

$\Leftrightarrow$ $\operatorname{null} T = \{0\}$

$\Leftrightarrow$ the corresponding homogeneous system

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \vdots \quad \ddots \quad \vdots \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{cases}$$ has no non-trivial solution

$\Leftrightarrow$ the column vectors of A are lin. indept

Thm 3.22 $\Rightarrow$

- number of variables $=$ nullity(A) $+$ rank(A)

Prop 3.23, 3.24 $\Rightarrow$

- If number of variables $>$ number of equations
  $(*)$ cannot have unique solution
- If number of variables $<$ number of equations
  $(*)$ does not have solution for some $\vec{b} \in \mathbb{R}^m$

Matices and Vector Space, Linear Maps

$\mathbb{F}^n$ $\longleftrightarrow$ (Finite dim) Vector spaces

Matrices $\longleftrightarrow$ Linear maps

Easy for computation

(eg. Gaussian elimination, determinant, diagonalization)

More General

Want to represent vector spaces and linear maps using matrices.

> **Defn** Let V be a finite dim vector space
>
> An ordered basis of V is a basis of V with a specific order on its elements.

eg The standard ordered basis of $\mathbb{F}^n$

$$\beta = \{e_1, e_2, \cdots, e_n\}$$

1st 2nd n-th

$M(T, \alpha, \beta) \in M_{m\times n}(\mathbb{F})$,

Rmk ① The notion of ordered basis is used in our reference book (Friedberg).

② In our textbook (Axler), a basis is a list of vectors $v_1, v_2, \cdots, v_n$, which already has a specific order.

With an ordered basis, we can represent a vector in a vector space by a column vector

> **Defn 3.62**
>
> Let $\alpha = \{v_1, \cdots, v_n\}$ be an ordered basis of V.
>
> Suppose $v \in V$. Then $\exists$ unique $c_1, \cdots, c_n \in \mathbb{F}$ such that $v = c_1 v_1 + \cdots + c_n v_n$.
>
> Define the matrix of v with respect to α to be
>
> $$M(v, \alpha) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in M_{n\times 1}(\mathbb{F}) \text{ or } \mathbb{F}^n$$

Similarly, for linear maps,

Defn 3.32 Let $T \in \mathcal{L}(V,W)$

$\alpha = \{v_1, \cdots v_n\}$ be an ordered basis of V.

$\beta = \{w_1, \cdots, w_m\}$ be an ordered basis of W,

Define the matrix of T with respect to α, β

to be $M(T,\alpha,\beta) \in M_{m\times n}(\mathbb{F})$, whose

i-th column is $M(T(v_i),\beta)$. i.e.

$$M(T,\alpha,\beta) = \begin{bmatrix} | & & | \\ M(T(v_1),\beta) & \cdots & M(T(v_n),\beta) \\ | & & | \end{bmatrix}$$

1st column   n-th column

Rmk If the choices of ordered bases are clear,

we write $M(v)$ for $M(v,\alpha)$, $M(T)$ for $M(T,\alpha,\beta)$

Notations in ref. book : $[v]_\alpha$, $[T]_\alpha^\beta$

eg Let $T: P_3(\mathbb{R}) \to P_2(\mathbb{R})$, $T(p(x)) = p'(x)$

$\alpha = \{1, x, x^2, x^3\}$, $\beta = \{1, x, x^2\}$ be basis

of $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$ respectively.

Then
$$\begin{aligned} T(1) &= 0 = 0\cdot 1 + 0\cdot x + 0x^2 \\ T(x) &= 1 = 1\cdot 1 + 0\cdot x + 0x^2 \\ T(x^2) &= 2x = 0\cdot 1 + 2x + 0x^2 \\ T(x^3) &= 3x^2 = 0\cdot 1 + 0x + 3x^2 \end{aligned}$$

$$M(T(1),\beta) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad M(T(x),\beta) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$M(T(x^2),\beta) = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \qquad M(T(x^3),\beta) = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

Hence $M(T,\alpha,\beta) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Formula Let V, W be finite dim, α, β be an ordered basis of V, W respectively, $v, v_1, v_2 \in V$, $T, T_1, T_2 \in L(V,W)$, $\lambda \in \mathbb{F}$. Then

① $M(v_1+v_2) = M(v_1)+M(v_2)$

② $M(\lambda v) = \lambda M(v)$

③ $M(T_1+T_2) = M(T_1)+M(T_2)$

④ $M(\lambda T) = \lambda M(T)$

⑤ $M(T(v)) = M(T)M(v)$

eg Use notation from last eg.

Let $S: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ $\quad S(p(x)) = p''(x)$

Then
$$M(S) = M(S,\alpha,\beta) = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore M(S+T) = M(S)+M(T) = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Also,
$$M((S+T)(1+x+x^2+x^3))$$
$$= M(S+T)\,M(1+x+x^2+x^3)$$
$$= \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 3 \end{bmatrix}$$

Check:
$$(S+T)(p(x)) = p''(x)+p'(x)$$
$$\therefore (S+T)(1+x+x^2+x^3)$$
$$= (1+2x+3x^2)+(2+6x)$$
$$= 3+8x+3x^2$$
(Agree with above)

Exercise Prove formula ① - ④ above.

We will prove ⑤

Recall:

Let $A \in M_{m\times n}(\mathbb{F})$ with columns $v_1, \cdots, v_n \in \mathbb{F}^m$

$B \in M_{n\times k}(\mathbb{F})$ with columns $w_1, \cdots, w_k \in \mathbb{F}^n$,

$$c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{F}^n$$

Then

$$Ac = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

$$AB = A \begin{bmatrix} | & | & & | \\ w_1 & w_2 & \cdots & w_k \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ Aw_1 & Aw_2 & \cdots & Aw_k \\ | & | & & | \end{bmatrix}$$

eg.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 4 \end{bmatrix} + (0)\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 16 & 15 \end{bmatrix}$$

Pf of 5: $M(T(v)) = M(T)M(v)$

Let $\alpha = \{v_1, \cdots, v_n\}$, $\beta = \{w_1, \cdots, w_m\}$ be ordered basis of V, W resp.

$T \in \mathcal{L}(V, W)$, $v \in V$. Suppose

$$v = c_1 v_1 + \cdots + c_n v_n$$

Then

$$\begin{aligned} & M(T(v)) \\ &= M(T(c_1 v_1 + \cdots + c_n v_n)) \\ &= M(c_1 T(v_1) + \cdots + c_n T(v_n)) \\ &= c_1 M(T(v_1)) + \cdots + c_n M(T(v_n)) \\ &= \begin{bmatrix} | & & | \\ M(T(v_1)) & \cdots & M(T(v_n)) \\ | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\ &= M(T)M(v) \end{aligned}$$

Prop 3.4.3 Let $T \in \mathcal{L}(U,V)$, $S \in \mathcal{L}(V,W)$

α, β, γ be ordered basis of U, V, W resp. Then

$$M(ST, \alpha, \gamma) = M(S, \beta, \gamma) M(T, \alpha, \beta)$$

i.e $M(ST) = M(S) M(T)$

Pf Let $\alpha = \{u_1, \cdots, u_k\}$

Then

$$\underbrace{M(ST(u_i))}_{\text{i-th column of } M(ST)} = M(S(T(u_i)))$$
$$= M(S) M(T(u_i))$$
$$= M(S) M(T) M(u_i)$$
$$= \underbrace{M(S)M(T)}_{\text{i-th column of } M(S)M(T)} e_i$$

$e_i \in \mathbb{F}^k$

$$\therefore \ M(ST) = M(S) M(T)$$

eg Let $\alpha = \{1, x, x^2, x^3\}$

$D : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be $Dp = p'$.

Then

$$M(D, \alpha) = M(D) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then

$$M(D^2) = M(D)^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M(D^2) = M(D)^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M(D^4) = M(D)^4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note $D^4 = T_0$ = zero transformation